Relativity and positioning systems

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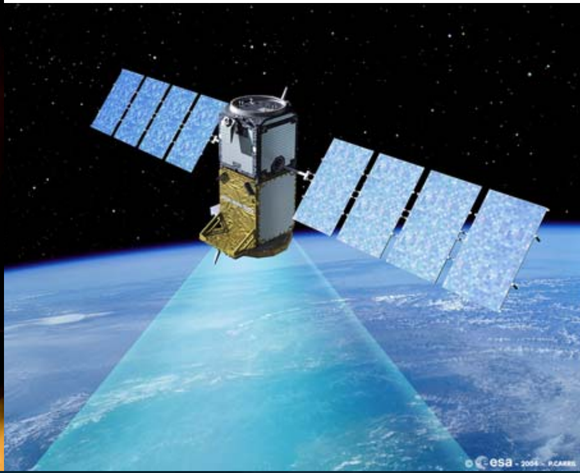
- **December 2005**: Giove-A -> secure the frequencies for the Galileo signals and validate the necessary technologies, such as the atomic clocks.

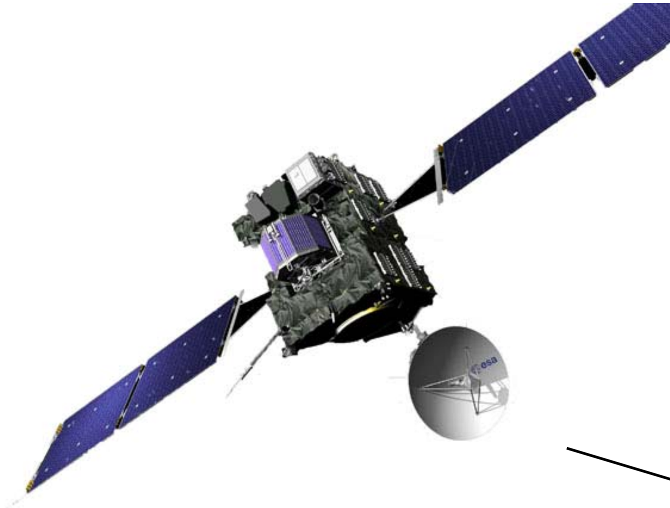
- **April 2008**: Giove-B launch

- **22 January 2010**: Gregor Golobič comes at the European Space Agency (ESA) to sign the European Cooperating State Agreement. The major participation should be in ESA's Science and Robotic Exploration programme.

- **26 January 2010**: ESA signs three contracts with industry to start the building of the Galileo operational infrastructure (Thales Alenia Space, OHB/SSTL, Arianespace) -> The first satellite is to be delivered in July 2012, and launched with a Soyuz rocket.



Basic principle of a positioning system (1/3)



- Atomic clock onboard the satellite -> send a signal to the receiver with the time of emission t_E
- Clock in the receiver -> time of the reception of the signal t_R

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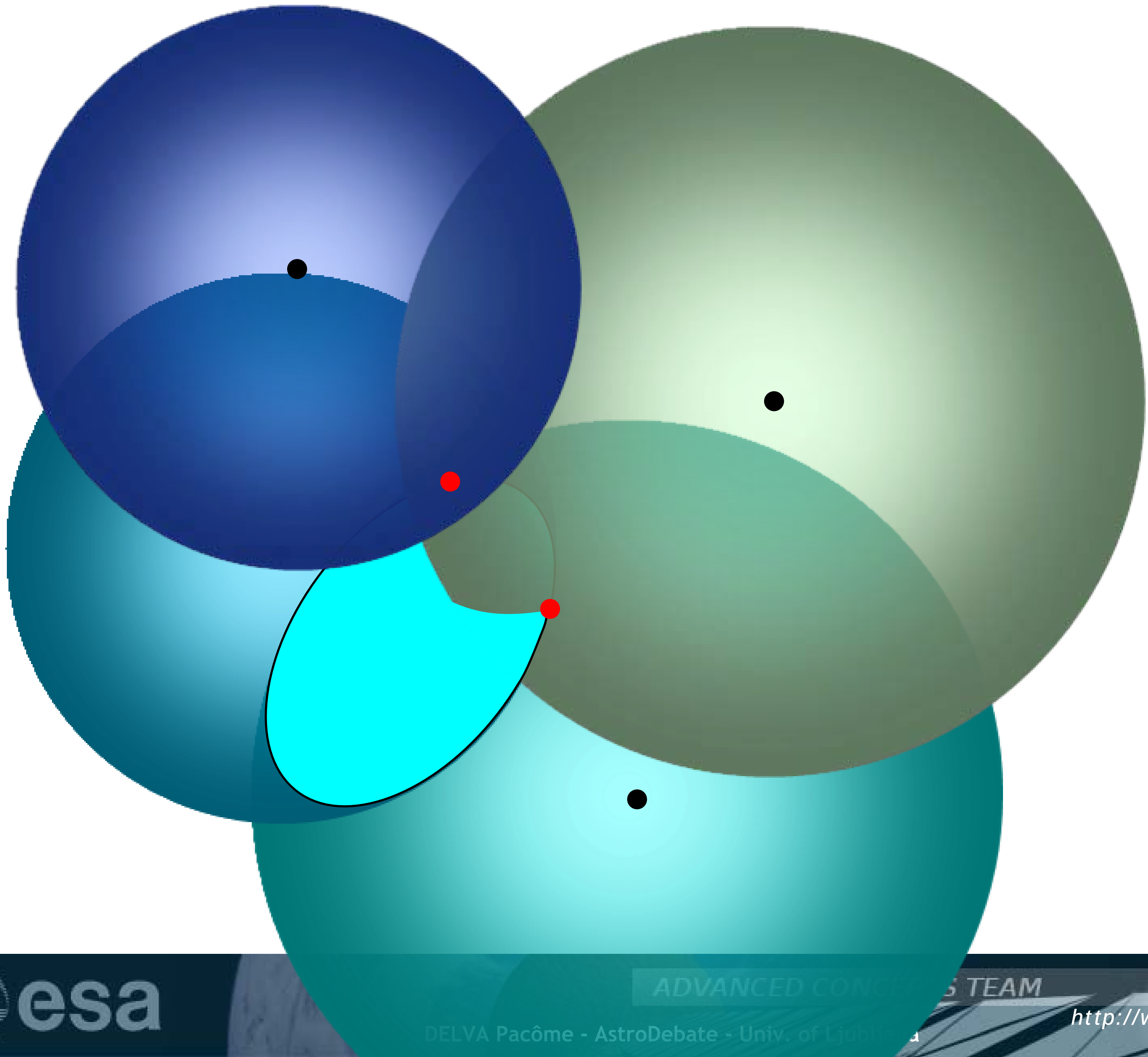
- The distance between the receiver and the satellite is:

$$(t_R - t_E)c$$

- We assume that the location of the satellite is known in a given coordinate system
- The receiver is located on the **surface of a sphere** of radius $(t_R - t_E)c$



- In term of equations… **three unknowns x,y,z** (position of the receiver) and **three equations** (if three satellites are in view) :

$$(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 = c^2(t_R - t_E)^2 \ , \ i = 1, 2, 3$$

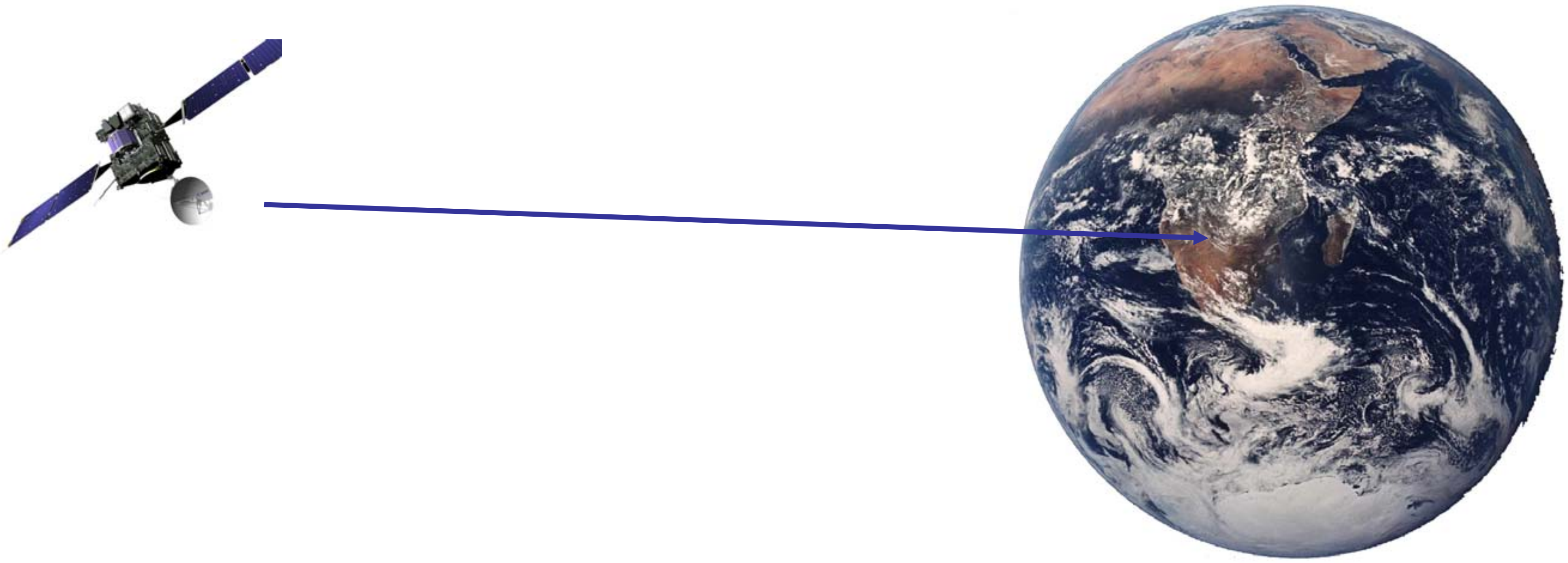
- This problem can be solved by **trilateration** and gives in practical cases two solutions

- Limitation in the precision comes usually from the bad receiver clock -> several methods to correct the receiver clock with four or more satellites

Why include relativity in the positioning system? (1/2)

- Relativity teach us that space and time are not absolute
- The GNSS is affected in three different ways:
  - In the **equations of motion** of the satellites
  - In the **signal propagation**
  - In the **beat rate** of the clocks
- In the future Galileo and the GPS, only the clock effects are measurable
- The most important ones are:
  - The **gravitational frequency shift** between the clocks (Equivalence principle of General Relativity)
  - The **Doppler shift of the second order** due to the motion of the satellites (Special Relativity)

- Gravitational frequency shift: **clocks run faster** when they are far away from a center of gravitational attraction
- Special relativistic Doppler effect: **clocks in motion slow down** (the satellite clocks move faster than the clock on Earth)
- These two effects are opposite and have a net blue shift effect -> equivalent to an error of ~12 km after one day of operation



Some order of magnitude (1/2)

- General Relativity is based upon the postulate of the **Einstein Equivalence Principle**, which can be be separated in three principles: the Weak Equivalence Principle, the Local Lorentz Invariance and the Local Position Invariance (LPI).
- LPI: any local (non-gravitational) experiment is independent of where and when in the in the universe it is performed ->

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 \mathrm{d}t^2 - \mathrm{d}r^2 - r^2 \mathrm{d}\varphi^2$$

- For a circular orbit $\mathrm{d}r = 0$ implies:

$$\frac{1}{c^2}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = \left(1 - \frac{2GM}{rc^2}\right) - \frac{v^2}{c^2} \qquad v = r\frac{\mathrm{d}\varphi}{\mathrm{d}t}$$

- We apply this formula to a clock in a satellite S with proper time $\mathrm{d}\tau_S = \mathrm{d}s/c$ and to a fixed clock R on Earth with proper time $\mathrm{d}\tau_R = \mathrm{d}s/c$:

$$\left(\frac{\mathrm{d}\tau_R}{\mathrm{d}\tau_S}\right)^2 = \frac{1 - \frac{2GM}{r_R c^2} - \frac{v_R^2}{c^2}}{1 - \frac{2GM}{r_S c^2} - \frac{v_S^2}{c^2}}$$

- In a first order approximation (weak gravitational field and low velocity) we obtain:

$$\frac{\mathrm{d}\tau_R}{\mathrm{d}\tau_S} = 1 - \frac{GM}{r_R c^2} - \frac{v_R^2}{2c^2} + \frac{GM}{r_S c^2} + \frac{v_S^2}{2c^2}$$

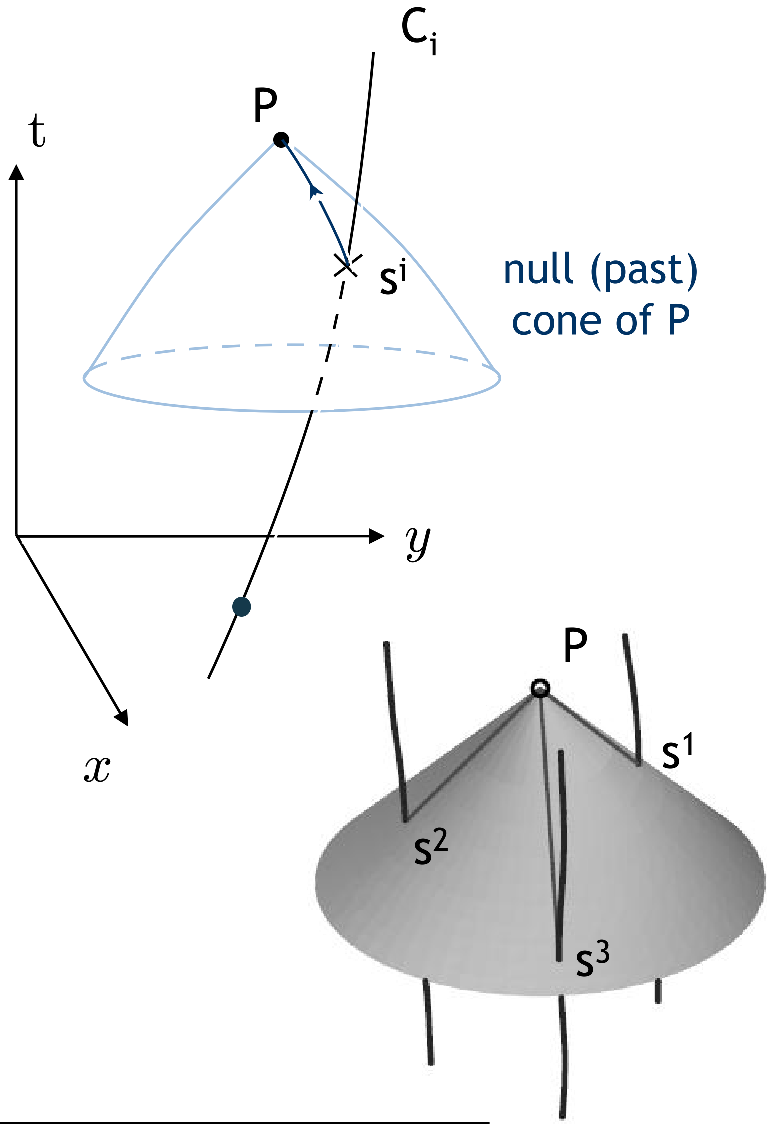
- For the Galileo constellation:

$$\begin{aligned} -\frac{GM}{r_R c^2} + \frac{GM}{r_S c^2} &= -5.4554 \cdot 10^{-10} \\ -\frac{v_R}{2c^2} + \frac{v_S}{2c^2} &= +7.3715 \cdot 10^{-11} \end{aligned}$$

- Which correspond to an error of ~ 12.2 km after one day of integration
- Question: why is the error for the GPS constellation, ~ 11.7 km, smaller ?

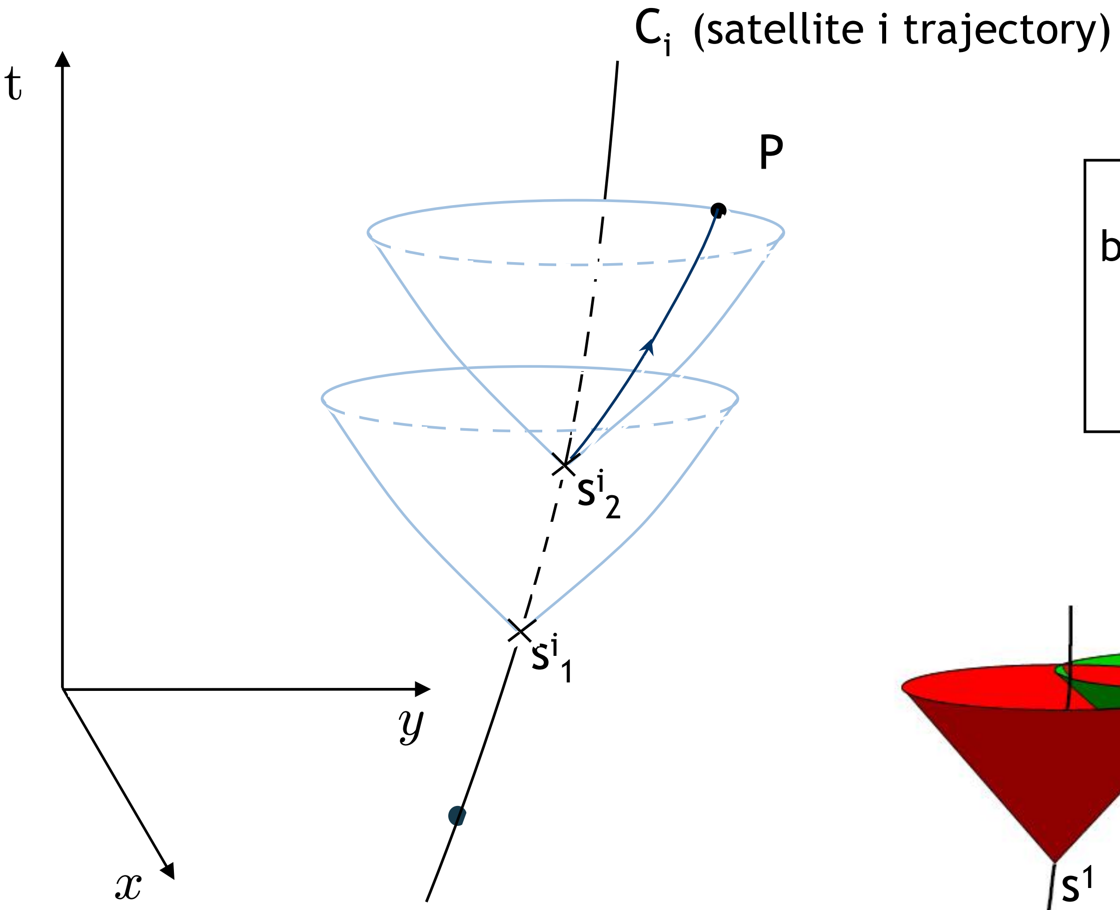
A necessary change of paradigm

- Practically, things are much more complicated: the GPS coordinate time is defined as the time of **a clock at rest on the geoid**
- The real calculation of the relativistic corrections involves transformation from the **ECEF** (Earth Centered Earth Fixed system) to the **ECI** (Earth Centered Inertial system), and includes corrections from the **quadrupole potential** of the Earth, the **kinematical Sagnac effect** due to the rotation of the ECEF, and the effect of the **eccentricity of the orbits**…
- The next generation of atomic clocks (optical clocks) will have a stability of a **few picoseconds** over one day. At this level, one has to take into account much more effects: secondary Sagnac effect due to the polar motion, tidal effects of the Moon and the Sun, the Shapiro time delay, and many other terms (see Linet *et al.*, PRD 66, 2002)
- Why this situation? Because the newtonian theory, even if known as the wrong theory to describe space and time, is still considered as the main theory and relativity as corrections
- A much simpler picture is given when starting the modelization of the constellation of satellites in a general relativistic framework



From Coll & Pozo, CQG 23 (2006)

- Consider General relativity coupled with four “test particles”, with an origin marked along the worldlines C_1, C_2, C_3 and C_4 of these particles
- A natural reference system in space-time can be defined:
  - Given a point P consider its past light cone. Usually, it intersect the worldline C_i once. Let (s^1,s^2,s^3,s^4) be the physical lengths of the portion of the worldlines between the origin and the intersection with the past light cone of P.
  - Then (s^1,s^2,s^3,s^4) are natural coordinates for P -> **null coordinates**



Reception of brodcasted proper time

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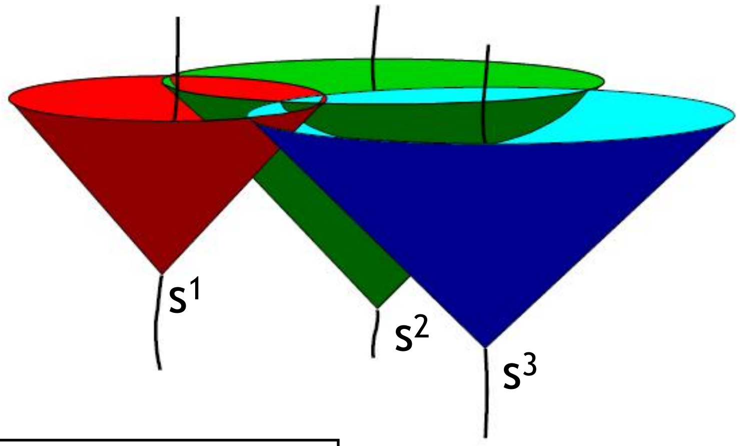
Null coordinate



From Coll & Pozo, CQG 23 (2006)

Some properties and advantages of the null coordinates

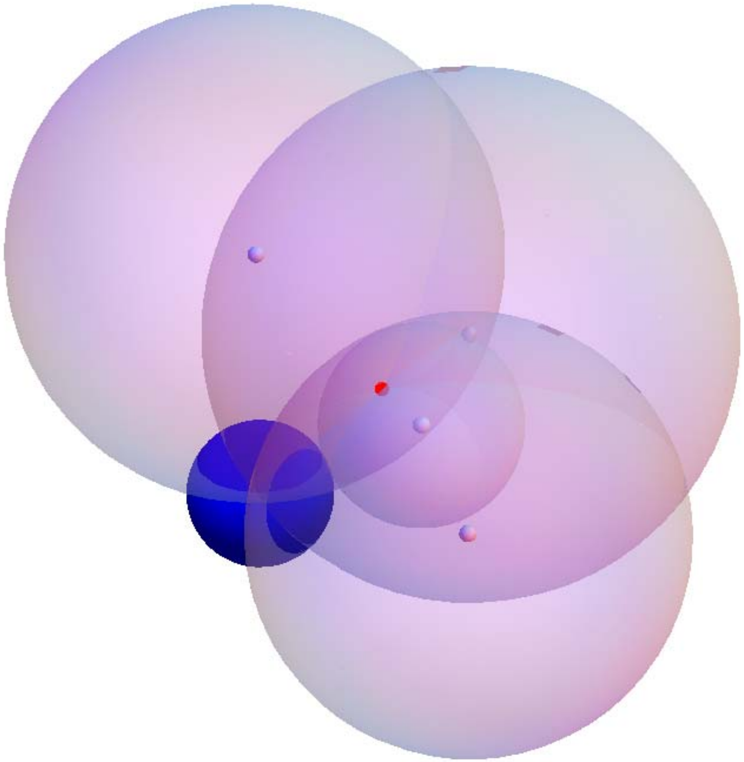
- Why null coordinates ?
  - In this coordinate system the metric has the form $g^{\mu\mu} = 0$
  - The four families of coordinate hypersurfaces $s^{\mu} = \mathrm{constant}$ are null hypersurfaces
- null coordinates are **covariant quantities** ie. they are independent of the observer: an observer at a given event in space-time will measure the same coordinates whatever is his velocity
- They define a **primary reference system**: no need for ground clocks (synchronization) as for the GPS and Galileo times -> **extra-terrestrial navigation** with pulsars as clocks
- Relativity is not seen as corrections to add: it is already included in the definition of the positioning system (no Doppler, gravitational shift, … corrections)
- But… They **depend on a set of satellites** and their trajectory;
- We are not used to this kind of coordinates (an observer at rest in space will not have constant null coordinates)
- However, they can be linked to usual coordinates via ground station on Earth -> the conceptual difficulties do not lie anymore in the definition of the primary reference frame but in the attempt to link the physically sounded relativistic reference frame to the abstract GPS reference frame.

- Collaboration between the University of Ljubljana (Prof. Čadež and Dr. Kostić) and the Advanced Concept Team of the European Space Agency.
- A first step towards the modelization of a GNSS in a relativistic framework and the understanding of a relativistic positioning system.
- The ingredients:
  - a background geometry: **Schwarzschild geometry** (in a first step);
  - the **equations of motion** of the satellites;
  - the **signal propagation;**
  - the **beat rate** of the clocks;
- We chose an **analytical method** to solve the set of non-linear differential equations, in terms of elliptical functions and integrals.
- Then the ingredients are used in a **numerical code** in order to give the coordinate transformations from the Schwarzschild coordinates to the null coordinates (and the inverse problem)
- Finally we study the effects of the non-gravitational perturbations (clocks errors, drag, micro-meteorites) on the positioning system

- Remember the equation of the positioning system in newtonian theory and add some noise on the receiver clock, assumed to be unknown -> you need a fourth satellite to correct for this additional unknown

$$t - t_i = T_f(\vec{x}, \vec{x}_i) = \frac{1}{c}\left[(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2\right]^{1/2} \ , \ i = 1,2,3,4$$

- Iterative method:



The time of flight in the Schwarzschild spacetime

- Differential equation for orbit of lightlike geodesics

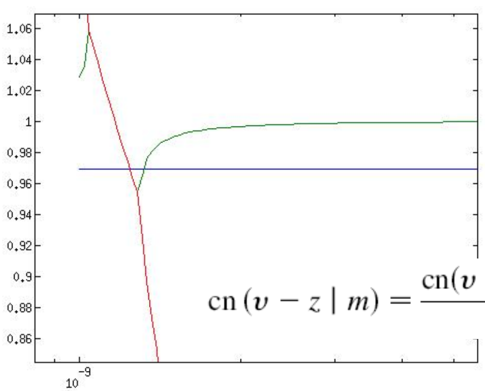
$$\frac{\mathrm{d}u}{\mathrm{d}\lambda} = \pm\sqrt{a^2 - u^2(1-u)} \qquad u = \frac{r_S}{r}$$

- Solution in terms of elliptical function and integral

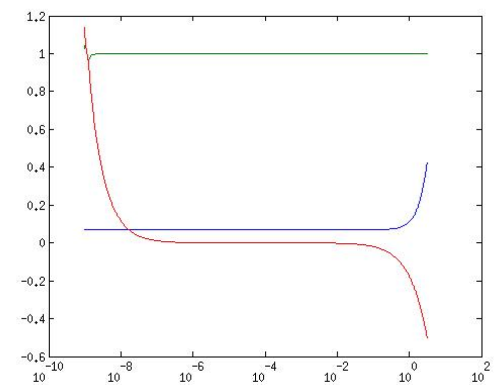
$$\mathrm{u}(\lambda) = u_2 - (u_2 - u_3)\mathrm{cn}^2\left[\left(F(\chi_A|m) + \frac{\lambda - \lambda_A}{n}\right)|m\right]$$

Čadež and Kostić, PRD 72 (2005)

- From the (light-like) geodesic and given two spatial point in Schwarzschild space-time, one can find the orbit parameter a by solving numerically the equation



$$\mathrm{cn}\,(v - z \mid m) = \frac{\mathrm{cn}(v \mid m)\mathrm{cn}(z \mid m) + \mathrm{sn}(v \mid m)\mathrm{sn}(z \mid m)\mathrm{dn}(v \mid m)\mathrm{dn}(z \mid m)}{1 - m\,\mathrm{sn}^2(v \mid m)\mathrm{sn}^2(z \mid m)}$$



- The time of flight T_f is then given as a function of the orbit parameter a

$$t - t_i(\tau^i) = T_f(\vec{x}, \vec{x}_i(\tau^i); a)$$

Conclusion

- Relativity has to be included in a positioning system → error of ~12 km after one day
- Up to now, relativity is added as perturbation to a newtonian conception of absolute time and space → simpler picture in a general relativy framewoark
- Four satellite define a primary (relativistic) reference frame, with the introduction of null coordinates → no need to tie this reference to the Earth → very stable reference frame (independent of Earth internal dynamics)
- Null coordinates are observer independent, and define a complete set of observables.
- We defined in a consistent way a relativistic positioning scheme using four or more satellites, and have shown that
- Need to go beyond the simplification of the Schwarzschild geometry by adding canonical perturbations to the metric → tidal effects of Moon and Sun, …

- The Metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$$

- Find the trajectories C_A of the 4 satellites (A=1,2,3,4)

$$\frac{\mathrm{d}^2 \boldsymbol{x_A}}{\mathrm{d}(\tau^A)^2} = 0 \quad \Rightarrow \quad \boldsymbol{x_A}(\tau^A) = \boldsymbol{U_A}\tau^A + \boldsymbol{S_A^0}$$

- Solve the time transfer equation

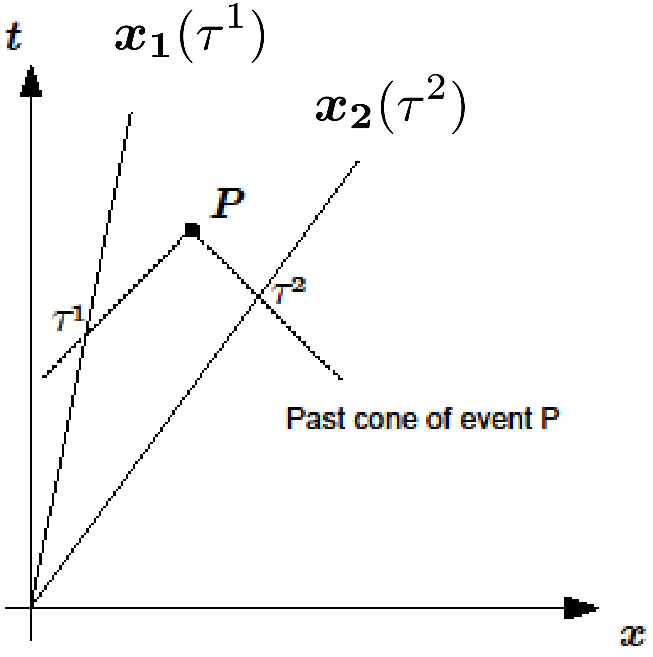
$$\Omega(\boldsymbol{x_A}(\tau^A), \boldsymbol{x_P}) = 0 \ , \ x_A^0(\tau^A) < x_P^0$$

$$\Downarrow$$

$$||\boldsymbol{x_P}||^2 - 2\tau^A \langle \boldsymbol{x_P}, \boldsymbol{U_A} \rangle + \left(\tau^A\right)^2 = 0$$

$$\Downarrow$$

$$\tau^A = \langle \boldsymbol{x_P}, \boldsymbol{U_A} \rangle - \sqrt{\langle \boldsymbol{x_P}, \boldsymbol{U_A} \rangle - ||\boldsymbol{x_P}||^2}$$



$$g^{AB} = \eta^{\alpha\beta} \frac{\partial \tau^A}{\partial x^\alpha} \frac{\partial \tau^B}{\partial x^\beta}$$